

Observables

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Ein etwas vorschnippischer Philosoph, ich glaube Hamlet, Prinz von Dänemark, hat gesagt, es gebe eine Menge Dinge im Himmel und auf der Erde, wovon nichts in unseren Compendiis steht. Hat der einfältige Mensch, der bekanntlich nicht recht bei Trost war, damit auf unsere Compendia der Physik gestichelt, so kann man ihm getrost antworten: Gut, aber dafür stehn auch wieder eine Menge von Dingen in unseren Compendiis, wovon weder im Himmel noch auf der Erde etwas vorkömmt.
 (Georg Christoph Lichtenberg, Aphorismen 1796 - 1799)

Is there any justification to speak nearly an hour about such a basic and apparently simple notion like “observable”? Are there some new - and interesting - insights, at least mathematically, or even conceptually?

I propose that this is the case.

Following Araki [1], an observable is an equivalence class of measuring instruments, two measuring instruments being equivalent if in any “state” of

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the “physical system” they lead upon a “large number of measurements” to the same distribution of (relative frequencies of) results. From this concept one can derive (a little bit in the sense of Ed Harris) that

- (i) in (von Neumann’s axiomatic approach to) quantum physics, an observable is represented by a bounded selfadjoint operator A acting on a Hilbert space \mathcal{H} , and
- (ii) in classical mechanics, an observable is represented by a real valued (smooth or continuous or measurable) function on an appropriate phase space.

Here a natural question arises: is the structural difference between classical and quantum observables fundamental, or is there some background structure showing that classical and quantum observables are on the same footing? Indeed, such a background structure exists, and I shall describe in this talk some of its features.

1 Motivation: Presheaves on a Lattice

A continuous classical observable is a continuous function $f : M \rightarrow \mathbb{R}$ on a (locally compact) Hausdorff space M . Equivalently, f can be considered as a global section of the presheaf \mathcal{C}_M of all real valued continuous functions that are defined on some nonempty open subsets of M . This situation leads to a natural generalization. The set $\mathcal{T}(M)$ of all open subsets of M can be seen as a *complete lattice*, the lattice operations being defined by

$$\bigvee_{k \in \mathbb{K}} U_k := \bigcup_{k \in \mathbb{K}} U_k, \quad \bigwedge_{k \in \mathbb{K}} U_k := \text{int}\left(\bigcap_{k \in \mathbb{K}} U_k\right).$$

(In the English language the word “lattice” has two different meanings. Either it is a subgroup of the additive group \mathbb{Z}^d for some $d \in \mathbb{N}$ (this is called “Gitter” in German) or it means a partially ordered set with certain additional properties. This is called “Verband” in German. We always use “lattice” in this second meaning.)

Definition 1.1 *A lattice is a partially ordered set (\mathbb{L}, \leq) such that any two elements $a, b \in \mathbb{L}$ possess a maximum $a \vee b \in \mathbb{L}$ and a minimum $a \wedge b \in \mathbb{L}$. Let \mathfrak{m} be an infinite cardinal number.*

The lattice \mathbb{L} is called \mathfrak{m} -complete, if every family $(a_i)_{i \in I}$ has a supremum $\bigvee_{i \in I} a_i$ and an infimum $\bigwedge_{i \in I} a_i$ in \mathbb{L} , provided that $\#I \leq \mathfrak{m}$ holds. A lattice

\mathbb{L} is simply called *complete*, if every family $(a_i)_{i \in I}$ in \mathbb{L} (without any restriction of the cardinality of I) has a supremum and an infimum in \mathbb{L} .

\mathbb{L} is said to be *boundedly complete* if every bounded family in \mathbb{L} has a supremum and an infimum.

If a lattice has a zero element 0 (i.e. $\forall a \in \mathbb{L} : 0 \leq a$) and a unit element 1 (i.e. $\forall a \in \mathbb{L} : a \leq 1$) then completeness and bounded completeness are the same.

A lattice \mathbb{L} is called *distributive* if the two distributive laws

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

hold for all elements $a, b, c \in \mathbb{L}$.

In fact it is an easy exercise to show that if one of these distributive laws is satisfied for all $a, b, c \in \mathbb{L}$ so is the other.

Traditionally, the notions of a **presheaf** and a **complete presheaf** (complete presheaves are usually called “sheaves”) are defined for the lattice $\mathcal{T}(M)$ of a topological space M . The very definition of presheaves and sheaves, however, can be formulated also for an arbitrary lattice:

Definition 1.2 A **presheaf** of sets (*R*-modules) on a lattice \mathbb{L} assigns to every element $a \in \mathbb{L}$ a set (*R*-module) $\mathcal{S}(a)$ and to every pair $(a, b) \in \mathbb{L} \times \mathbb{L}$ with $a \leq b$ a mapping (*R*-module homomorphism)

$$\rho_a^b : \mathcal{S}(b) \rightarrow \mathcal{S}(a)$$

such that the following two properties hold:

- (1) $\rho_a^a = \text{id}_{\mathcal{S}(a)}$ for all $a \in \mathbb{L}$,
- (2) $\rho_a^b \circ \rho_b^c = \rho_a^c$ for all $a, b, c \in \mathbb{L}$ such that $a \leq b \leq c$.

The presheaf $(\mathcal{S}(a), \rho_a^b)_{a \leq b}$ is called a **complete presheaf** (or a **sheaf** for short) if it has the additional property

- (3) If $a = \bigvee_{i \in I} a_i$ in \mathbb{L} and if $f_i \in \mathcal{S}(a_i)$ ($i \in I$) are given such that

$$\forall i, j \in I : (a_i \wedge a_j \neq 0 \implies \rho_{a_i \wedge a_j}^{a_i}(f_i) = \rho_{a_i \wedge a_j}^{a_j}(f_j),$$

then there is exactly one $f \in \mathcal{S}(a)$ such that

$$\forall i \in I : \rho_{a_i}^a(f) = f_i.$$

The mappings $\rho_a^b : \mathcal{S}(b) \rightarrow \mathcal{S}(a)$ are called **restriction maps**.

Concerning quantum theory, the fundamental lattice is the lattice $\mathbb{L}(\mathcal{H})$ of all *closed* linear subspaces of a Hilbert space \mathcal{H} . This lattice is defined by

- (i) $U \leq V \iff U \subseteq V$,
- (ii) $\bigwedge_{k \in \mathbb{K}} U_k := \bigcap_{k \in \mathbb{K}} U_k$,
- (iii) $\bigvee_{k \in \mathbb{K}} U_k := \overline{\sum_{k \in \mathbb{K}} U_k}$.

There are interesting presheaves on $\mathbb{L}(\mathcal{H})$ but, unfortunately, any complete presheaf on $\mathbb{L}(\mathcal{H})$ is also completely trivial:

Proposition 1.1 *Let $(\mathcal{S}(U), \rho_U^V)_{U \subseteq V}$ be a complete presheaf of nonempty sets on the quantum lattice $\mathbb{L}(\mathcal{H})$. Then*

$$\#\mathcal{S}(U) = 1$$

for all $U \in \mathbb{L}(\mathcal{H})$.

In order to present a fundamental example of a presheaf on $\mathbb{L}(\mathcal{H})$ (which turns out to be rather universal), we make the following general

Definition 1.3 *Let \mathbb{L} be a complete lattice. A mapping*

$$\begin{aligned} E : \mathbb{R} &\rightarrow \mathbb{L} \\ \lambda &\mapsto E_\lambda \end{aligned}$$

is called a **spectral family** in \mathbb{L} , if it has the following properties:

- (i) $E_\lambda \leq E_\mu$ for $\lambda \leq \mu$,
- (ii) $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu$ for all $\lambda \in \mathbb{R}$,
- (iii) $\bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0$, $\bigvee_{\lambda \in \mathbb{R}} E_\lambda = 1$.

A spectral family E in \mathbb{L} is called *bounded* if there are $a, b \in \mathbb{R}$ such that $E_\lambda = 0$ for all $\lambda < a$ and $E_\lambda = 1$ for all $\lambda \geq b$.

If \mathbb{L} is a complete lattice and $a \in \mathbb{L}$, then

$$\mathbb{L}_a := \{b \in \mathbb{L} \mid b \leq a\}$$

is a complete sublattice with maximal element a . Let E be a bounded spectral family in \mathbb{L} . Then

$$E^a : \lambda \mapsto E_\lambda \wedge a$$

is a spectral family in \mathbb{L}_a . E^a is called the *restriction of E to a* . Let $\mathcal{E}(a)$ be the set of all spectral families in \mathbb{L}_a . Then for $a, b \in \mathbb{L}$, $a \leq b$, we obtain a mapping

$$\begin{array}{ccc} \varrho_{ab} : \mathcal{E}(b) & \rightarrow & \mathcal{E}(a) \\ E & \mapsto & E^a, \end{array}$$

and it is easy to see that the sets $\mathcal{E}(a)$ together with the restriction maps ϱ_{ab} form a presheaf on \mathbb{L} . We denote this presheaf by $\mathcal{E}_{\mathbb{L}}$ and call it the *spectral presheaf on \mathbb{L}* . Let us give a “classical” example:

Example 1.1 *Let $f : M \rightarrow \mathbb{R}$ be a continuous function on a Hausdorff space M . Then*

$$\forall \lambda \in \mathbb{R} : E_\lambda := \text{int}(f^{-1}([-\infty, \lambda]))$$

defines a spectral family $E : \mathbb{R} \rightarrow \mathcal{T}(M)$. (The natural guess for defining a spectral family corresponding to f would be

$$\lambda \mapsto f^{-1}([-\infty, \lambda]).$$

In general, this is only a pre-spectral family: it satisfies all properties of a spectral family, except continuity from the right. This is cured by spectralization, i.e. by the switch to

$$\lambda \mapsto \bigcap_{\mu > \lambda} f^{-1}([-\infty, \mu]).$$

But

$$\bigcap_{\mu > \lambda} f^{-1}([-\infty, \mu]) = \text{int}\left(\bigcap_{\mu > \lambda} f^{-1}([-\infty, \mu])\right) = \text{int}(f^{-1}([-\infty, \lambda])),$$

which shows that our original definition is the natural one.)

One can show that

$$\forall x \in M : f(x) = \inf\{\lambda \mid x \in E_\lambda\},$$

so one can recover the function f from its spectral family E . Let $U \in \mathcal{T}(M)$, $U \neq \emptyset$. Then

$$E_\lambda \cap U = \text{int}\{x \in U \mid f(x) \leq \lambda\} = \text{int}(f|_U^{-1}([-\infty, \lambda])),$$

hence E^U is the spectral family of the usual restriction $f|_U : U \rightarrow \mathbb{R}$ of f to U .

Another example comes from the theory of operator algebras:

Example 1.2 *Let \mathcal{R} be a von Neumann algebra, acting on a Hilbert space \mathcal{H} , let \mathcal{R}_{sa} be the set of selfadjoint operators in \mathcal{R} and let $\mathcal{P}(\mathcal{R})$ be the lattice of projections in \mathcal{R} . The spectral family E of $A \in \mathcal{R}_{sa}$ is a spectral family in $\mathcal{P}(\mathcal{R})$ and therefore the restriction*

$$E^P : \lambda \mapsto E_\lambda \wedge P$$

of E to $P \in \mathcal{P}(\mathcal{R})$ is a bounded spectral family in the ideal

$$I_P := \{Q \in \mathcal{P}(\mathcal{R}) \mid Q \leq P\} \subseteq \mathcal{P}(\mathcal{R}).$$

It is easy to see that the operator A^P corresponding to E^P belongs to PRP , a von Neumann algebra operating on the Hilbert space $P\mathcal{H}$. In particular, A^P can be considered as an element of $\mathcal{L}(P\mathcal{H})$. Note that, although $A^P \in PRP$, A^P is, in general, different from PAP . This follows from the fact that A^P is a projection if A is, but that, for a projection A , PAP is a projection if and only if A commutes with P .

For $P \in \mathcal{P}(\mathcal{R})$, let $\mathcal{E}(P)$ be the set of bounded spectral families in I_P , and for $P, Q \in \mathcal{P}(\mathcal{R})$ such that $P \leq Q$, we define a restriction map

$$\begin{array}{ccc} \varrho_{PQ} : \mathcal{E}(Q) & \rightarrow & \mathcal{E}(P) \\ E & \mapsto & E^P. \end{array}$$

It is obvious that the sets $\mathcal{E}(P)$ together with the restriction maps ϱ_{PQ} form a presheaf $\mathcal{E}_{\mathcal{R}}$ on the projection lattice $\mathcal{P}(\mathcal{R})$.

In general, one cannot drop the assumption that the spectral families are bounded, for it may happen that, if E is not bounded from above, $\bigvee_{\lambda \in \mathbb{R}} E_\lambda^P \neq P$ holds. More precisely, one can show ([2]) that E^P is a spectral family for all $P \in \mathcal{P}(\mathcal{R})$ and any spectral family E in $\mathcal{P}(\mathcal{R})$ if and only if \mathcal{R} is a finite von Neumann algebra.

It is well known that one can associate to each preasheaf \mathcal{S}_M on a topological space M a sheaf on M in the following way:

If \mathcal{S} is a presheaf on a topological space M , i.e. on the lattice $\mathcal{T}(M)$, then the corresponding *etale space* $\mathcal{E}(\mathcal{S})$ of \mathcal{S} is the disjoint union of the *stalks* of \mathcal{S} at points in M :

$$\mathcal{E}(\mathcal{S}) = \coprod_{x \in M} \mathcal{S}_x$$

where

$$\mathcal{S}_x = \varinjlim_{U \ni x} \mathcal{S}(U),$$

the *inductive limit* of the family $(\mathcal{S}(U))_{U \in \mathfrak{U}(x)}$, (here $\mathfrak{U}(x)$ denotes the set of all open neighbourhoods of x) is the stalk in $x \in M$. The stalk \mathcal{S}_x consists of the *germs* in x of elements $f \in \mathcal{S}(U)$, $U \in \mathfrak{U}(x)$. Germs are defined quite analogously to the case of ordinary functions. Let $\pi : \mathcal{E}(\mathcal{S}) \rightarrow M$ be the mapping that sends a germ in x to its basepoint x . $\mathcal{E}(\mathcal{S})$ can be given a topology for which π is a local homeomorphism. It is easy to see that the local sections of π form a complete presheaf on M . If \mathcal{S} was already complete, then this presheaf of local sections of π is isomorphic to \mathcal{S} .

A first attempt to generalize this construction to the situation of a presheaf on a general lattice \mathbb{L} is to define a suitable notion of “point in a lattice”. This can be done in a quite natural manner, and it turns out that, for *regular* topological spaces M , the points in $\mathcal{T}(M)$ are of the form $\mathfrak{U}(x)$, hence correspond to the elements of M . But it also turns out that some important lattices, like $\mathbb{L}(\mathcal{H})$, do not have points at all ([2])!

For the definition of an inductive limit, however, we do not need a point, like $\mathfrak{U}(x)$, but only a partially ordered set I with the property

$$\forall \alpha, \beta \in I \quad \exists \gamma \in I : \gamma \leq \alpha \quad \text{and} \quad \gamma \leq \beta.$$

In other words: a *filter base* B in a lattice \mathbb{L} is sufficient. It is obvious how to define a filter base in an arbitrary lattice \mathbb{L} :

Definition 1.4 *A filter base B in a lattice \mathbb{L} is a non-empty subset $B \subseteq \mathbb{L}$ such that*

- (1) $0 \notin B$,
- (2) $\forall a, b \in B \quad \exists c \in B : c \leq a \wedge b$.

The set of all filter bases in a lattice \mathbb{L} is of course a vast object. So it is reasonable to consider *maximal* filter bases in \mathbb{L} . (By Zorn’s lemma, every filter base is contained in a maximal filter base in \mathbb{L} .) This leads to the following

Definition 1.5 *A nonempty subset \mathfrak{B} of a lattice \mathbb{L} is called a **quasipoint** in \mathbb{L} if and only if*

- (1) $0 \notin \mathfrak{B}$,
- (2) $\forall a, b \in \mathfrak{B} \quad \exists c \in \mathfrak{B} : c \leq a \wedge b$,
- (3) \mathfrak{B} is a maximal subset having the properties (1) and (2).

We denote the set of quasipoints in \mathbb{L} by $\mathcal{Q}(\mathbb{L})$.

It is easy to see that a quasipoint in \mathbb{L} is nothing else but a *maximal dual ideal*.

In 1936 M.H.Stone ([6]) showed that the set $\mathcal{Q}(\mathcal{B})$ of quasipoints in a Boolean algebra \mathcal{B} can be given a topology such that $\mathcal{Q}(\mathcal{B})$ is a *compact zero dimensional* Hausdorff space and that the Boolean algebra \mathcal{B} is isomorphic to the Boolean algebra of all *closed open* subsets of $\mathcal{Q}(\mathcal{B})$. A basis for this topology is simply given by the sets

$$\mathcal{Q}_U(\mathcal{B}) := \{\mathfrak{B} \in \mathcal{Q}(\mathcal{B}) \mid U \in \mathfrak{B}\}$$

where U is an arbitrary element of \mathcal{B} .

Of course we can generalize this construction to an arbitrary lattice \mathbb{L} :
For $a \in \mathbb{L}$ let

$$\mathcal{Q}_a(\mathbb{L}) := \{\mathfrak{B} \in \mathcal{Q}(\mathbb{L}) \mid a \in \mathfrak{B}\}.$$

It is quite obvious from the definition of a quasipoint that

$$\mathcal{Q}_{a \wedge b}(\mathbb{L}) = \mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_b(\mathbb{L}),$$

$$\mathcal{Q}_0(\mathbb{L}) = \emptyset \quad \text{and} \quad \mathcal{Q}_I(\mathbb{L}) = \mathcal{Q}(\mathbb{L})$$

hold. Hence $\{\mathcal{Q}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$ is a basis for a topology on $\mathcal{Q}(\mathbb{L})$. It is easy to see, using the maximality of quasipoints, that in this topology the sets $\mathcal{Q}_a(\mathbb{L})$ are open and closed. Moreover, this topology is Hausdorff, zero-dimensional, and therefore also completely regular.

Definition 1.6 $\mathcal{Q}(\mathbb{L})$, together with the topology defined by the basis $\{\mathcal{Q}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$, is called the **Stone spectrum of the lattice \mathbb{L}** .

Then we can mimic the construction of the etale space of a presheaf on a topological space M to obtain from a presheaf \mathcal{S} on a lattice \mathbb{L} an etale space $\mathcal{E}(\mathcal{S})$ over the Stone spectrum $\mathcal{Q}(\mathbb{L})$ and a local homeomorphism $\pi_{\mathcal{S}} : \mathcal{E}(\mathcal{S}) \rightarrow \mathcal{Q}(\mathbb{L})$. From the etale space $\mathcal{E}(\mathcal{S})$ over $\mathcal{Q}(\mathbb{L})$ we obtain a complete presheaf $\mathcal{S}^{\mathcal{Q}}$ on the topological space $\mathcal{Q}(\mathbb{L})$ by

$$\mathcal{S}^{\mathcal{Q}}(\mathcal{V}) := \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$$

where $\mathcal{V} \subseteq \mathcal{Q}(\mathbb{L})$ is an open set and $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$ is the set of **sections of $\pi_{\mathcal{S}}$ over \mathcal{V}** , i.e. of all (necessarily continuous) mappings $s_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{E}(\mathcal{S})$ such that $\pi_{\mathcal{S}} \circ s_{\mathcal{V}} = id_{\mathcal{V}}$. If \mathcal{S} is a presheaf of modules, then $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$ is a module, too.

Definition 1.7 *The complete presheaf $\mathcal{S}^{\mathcal{Q}}$ on the Stone spectrum $\mathcal{Q}(\mathbb{L})$ is called the **sheaf associated to the presheaf \mathcal{S} on \mathbb{L}** .*

Of course, Stone had quite another motivation for introducing the space $\mathcal{Q}(\mathcal{B})$ of a Boolean algebra \mathcal{B} , namely to represent \mathcal{B} as a Boolean algebra of sets. The remarkable fact is that we arrive at a generalization of Stone's concept from a completely different point of view.

Let us return to example 1.2 and consider the special case $\mathcal{R} = \mathcal{L}(\mathcal{H})$. If $x \in \mathcal{H}$ is a unit vector, $P_{\mathbb{C}x}$ the orthogonal projection onto the line $\mathbb{C}x$, $A \in \mathcal{L}(\mathcal{H})$ is selfadjoint and if E is its spectral family, then the restriction $E^{P_{\mathbb{C}x}}$ of E to $P_{\mathbb{C}x}$ corresponds to a linear operator

$$A^{P_{\mathbb{C}x}} : \mathbb{C}x \rightarrow \mathbb{C}x$$

which is a scalar multiple $cI_{\mathbb{C}x}$ of the identity $I_{\mathbb{C}x}$. Because $\mathcal{P}(\mathbb{C}x) = \{0, P_{\mathbb{C}x}\}$, the spectral family $E^{P_{\mathbb{C}x}}$ has the form

$$E_{\lambda}^{P_{\mathbb{C}x}} = \begin{cases} 0 & \text{for } \lambda < c \\ P_{\mathbb{C}x} & \text{for } \lambda \geq c. \end{cases}$$

It is obvious from the definition of restriction that

$$c = \inf\{\lambda \mid P_{\mathbb{C}x} \leq E_{\lambda}\}. \quad (1)$$

This is the right place to report some results on Stone spectra of the projection lattice of a von Neumann algebra.

If \mathcal{R} is an abelian von Neumann algebra, then \mathcal{R} is $*$ -isomorphic to the von Neumann algebra $C(\Omega)$, the algebra of all complex-valued continuous functions on the *Gelfand spectrum* Ω of \mathcal{R} . Ω is the set of all multiplicative positive linear functionals on \mathcal{R} . It is a compact space with respect to the weak*-topology.

Theorem 1.1 *Let \mathcal{R} be an abelian von Neumann algebra. Then the Gelfand spectrum Ω is homeomorphic to the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of \mathcal{R} .*

The proof is based on the simple observation that, for every $\tau \in \Omega$,

$$\beta(\tau) := \{P \in \mathcal{P}(\mathcal{R}) \mid \tau(P) = 1\}$$

is a quasipoint in $\mathcal{P}(\mathcal{R})$. The mapping $\tau \mapsto \beta(\tau)$ turns out to be the assured homeomorphism.

Proposition 1.2 *Let \mathcal{H} be a Hilbert space and let \mathfrak{B} be a quasipoint in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. \mathfrak{B} contains an element of finite rank if and only if there is a (necessarily unique) line $\mathbb{C}x$ in \mathcal{H} such that*

$$\mathfrak{B} = \{P \in \mathcal{P}(\mathcal{L}(\mathcal{H})) \mid P_{\mathbb{C}x} \leq P\}.$$

If \mathcal{H} has infinite dimension, then \mathfrak{B} does not contain an element of finite rank if and only if $P \in \mathfrak{B}$ for all $P \in \mathcal{P}(\mathcal{L}(\mathcal{H}))$ of finite corank.

Quasipoints of the form $\{P \in \mathcal{P}(\mathcal{L}(\mathcal{H})) \mid P_{\mathbb{C}x} \leq P\}$ in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ are called *atomic*. These are precisely the *isolated points* of the topological space $\mathcal{Q}(\mathcal{H})$. They form a dense subset of $\mathcal{Q}(\mathcal{H})$.

Let \mathcal{R} be an arbitrary von Neumann algebra with center \mathcal{C} . It can be shown that, for every $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, the intersection $\mathfrak{B} \cap \mathcal{C}$ is a quasipoint in \mathcal{C} , and that the mapping

$$\begin{aligned} \zeta : \mathcal{Q}(\mathcal{R}) &\rightarrow \mathcal{Q}(\mathcal{C}) \\ \mathfrak{B} &\mapsto \mathfrak{B} \cap \mathcal{C} \end{aligned}$$

is surjective, continuous and open.

The unitary group $\mathfrak{U}_{\mathcal{R}}$ of \mathcal{R} acts in a natural way on the Stone spectrum $\mathcal{Q}(\mathcal{R})$:

$$T.\mathfrak{B} := \{T^*PT \mid P \in \mathfrak{B}\} \in \mathcal{Q}(\mathcal{R})$$

for all $T \in \mathfrak{U}_{\mathcal{R}}$ and all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$.

Theorem 1.2 ([4]) *Let \mathcal{R} be a finite von Neumann algebra of type I and let \mathcal{C} be the center of \mathcal{R} . Then the orbits of the action of the unitary group $\mathfrak{U}_{\mathcal{R}}$ of \mathcal{R} on $\mathcal{Q}(\mathcal{R})$ are the fibres of ζ .*

The characterization of the Stone spectra for other types of von Neumann algebras seems to be a really hard problem.

Note that equation (1) can be rewritten as

$$c = \inf\{\lambda \mid E_{\lambda} \in \mathfrak{B}_{P_{\mathbb{C}x}}\},$$

where $\mathfrak{B}_{P_{\mathbb{C}x}}$ is the atomic quasipoint determined by $P_{\mathbb{C}x}$. This gives a function on the set of atomic quasipoints of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ which has a natural extension to the whole Stone spectrum $\mathcal{Q}(\mathcal{H})$:

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{H}) : f_A(\mathfrak{B}) := \inf\{\lambda \mid E_{\lambda} \in \mathfrak{B}\}.$$

Of course this can be generalized to arbitrary von Neumann algebras and even to arbitrary complete lattices:

Definition 1.8 Let $E : \mathbb{R} \rightarrow \mathbb{L}$ be a bounded spectral family in a complete lattice \mathbb{L} . Then the function

$$\begin{aligned} f_E : \mathcal{Q}(\mathbb{L}) &\rightarrow \mathbb{R} \\ \mathfrak{B} &\mapsto \inf\{\lambda \mid E_\lambda \in \mathfrak{B}\} \end{aligned}$$

is called the **observable function** corresponding to E .

2 Observable Functions

In this section we discuss quantum and classical observables from a new perspective.

2.1 Quantum Observables

In what follows, \mathcal{R} is a von Neumann algebra acting on a Hilbert space \mathcal{H} . \mathcal{R}_{sa} denotes the set of all selfadjoint elements of \mathcal{R} . If $A \in \mathcal{R}_{sa}$ with spectral family E , then the observable function $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ is defined by

$$f_A(\mathfrak{B}) := \inf\{\lambda \mid E_\lambda \in \mathfrak{B}\}.$$

$sp(A)$ denotes the spectrum of A .

2.1.1 Basic Properties

The first basic property of observable functions is

Theorem 2.1 Let $A \in \mathcal{R}_{sa}$ and let $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ be the observable function corresponding to A . Then

$$im f_A = sp(A).$$

The proof rests on the fact that the spectrum $sp(A)$ of A consists of all $\lambda \in \mathbb{R}$ such that the spectral family E of A is non-constant on every neighbourhood of λ .

Example 2.1 The observable function of a projection $P \in \mathcal{P}(\mathcal{R})$ is given by

$$f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})},$$

where $\chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$ denotes the characteristic function of the open closed set $\mathcal{Q}_{I-P}(\mathcal{R})$. Hence f_P is a continuous function.

If \mathcal{R} is abelian then

$$f_P = \chi_{\mathcal{Q}_P(\mathcal{R})}.$$

This example is not accidental:

Theorem 2.2 *Let $A \in \mathcal{R}_{sa}$. Then the observable function $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ is continuous.*

The proof requires some work: we know from the spectral theorem that A can be approximated in norm by finite linear combinations of pairwise orthogonal projections. Explicit calculation of the observable function of such a linear combination shows that it is continuous. One proves eventually that the observable functions of the approximating operators converge uniformly to f_A .

Definition 2.1 *Let \mathcal{R} be a von Neumann algebra. Then we denote by $\mathcal{O}(\mathcal{R})$ the set of observable functions $\mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$.*

By the foregoing result $\mathcal{O}(\mathcal{R})$ is a subset of $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, the algebra of all bounded continuous functions $\mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$. $\mathcal{O}(\mathcal{R})$ separates the points of $\mathcal{Q}(\mathcal{R})$ because the observable function of a projection P is $f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$. Moreover it contains the constant functions. In general, however, it is not an algebra and not even a vector space (with respect to the pointwise defined algebraic operations).

Theorem 2.3 *Let \mathcal{R} be a von Neumann algebra and let $\mathcal{O}(\mathcal{R})$ be the set of observable functions on $\mathcal{Q}(\mathcal{R})$. Then*

$$\mathcal{O}(\mathcal{R}) = C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$$

if and only if \mathcal{R} is abelian.

The fact that $\mathcal{O}(\mathcal{R}) = C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$ for abelian \mathcal{R} (note that $\mathcal{Q}(\mathcal{R})$ is compact in this case) has a deeper reason:

Theorem 2.4 *Let \mathcal{A} be an abelian von Neumann algebra. Then the mapping $A \mapsto f_A$ from \mathcal{A} onto $C(\mathcal{Q}(\mathcal{A}), \mathbb{R})$ is, up to the homeomorphism of theorem 1.1, the restriction of the Gelfand transformation to \mathcal{A}_{sa} .*

This result shows that the concept of “observable function” is a noncommutative extension of the concept of “Gelfand transform”.

Here a natural question arises: Can we characterize abstractly those elements of $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$ that are observable functions? An “abstract characterization” of an observable function f_A should be a set of properties of f_A in which the operator A does not occur explicitly.

2.1.2 Abstract Characterization of Observable Functions

Let A be a selfadjoint element of a von Neumann algebra \mathcal{R} . The definition of the observable function $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$,

$$f_A(\mathfrak{B}) := \inf\{\lambda \mid E_\lambda \in \mathfrak{B}\},$$

can be extended without any change to the space of *all* dual ideals in $\mathcal{P}(\mathcal{R})$.

Definition 2.2 *Let \mathbb{L} be a complete lattice (with minimal element 0 and maximal element 1). A nonempty subset $\mathcal{J} \subseteq \mathbb{L}$ is called a **dual ideal** if it has the following properties:*

- (i) $0 \notin \mathcal{J}$,
- (ii) $a, b \in \mathcal{J} \implies a \wedge b \in \mathcal{J}$,
- (iii) if $a \in \mathcal{J}$ and $a \leq b$ then $b \in \mathcal{J}$.

If $a \in \mathbb{L} \setminus \{0\}$ then the dual ideal

$$H_a := \{b \in \mathbb{L} \mid b \geq a\}$$

is called the **principal dual ideal generated by a** . We denote by $\mathcal{D}(\mathbb{L})$ the set of dual ideals of \mathbb{L} . For $a \in \mathbb{L}$ let

$$\mathcal{D}_a(\mathbb{L}) := \{\mathcal{J} \in \mathcal{D}(\mathbb{L}) \mid a \in \mathcal{J}\}.$$

As mentioned earlier a maximal dual ideal is nothing but a quasipoint of \mathbb{L} . We collect some obvious properties of the sets $\mathcal{D}_a(\mathbb{L})$ in the following

Remark 2.1 *For all $a, b \in \mathbb{L}$ the following properties hold:*

- (i) $a \leq b \implies \mathcal{D}_a(\mathbb{L}) \subseteq \mathcal{D}_b(\mathbb{L})$,
- (ii) $\mathcal{D}_{a \wedge b}(\mathbb{L}) = \mathcal{D}_a(\mathbb{L}) \cap \mathcal{D}_b(\mathbb{L})$,
- (iii) $\mathcal{D}_a(\mathbb{L}) \cup \mathcal{D}_b(\mathbb{L}) \subseteq \mathcal{D}_{a \vee b}(\mathbb{L})$,
- (iv) $\mathcal{D}_0(\mathbb{L}) = \emptyset$, $\mathcal{D}_1(\mathbb{L}) = \mathcal{D}(\mathbb{L})$.

These properties show in particular that $\{\mathcal{D}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$ is a basis of a topology on $\mathcal{D}(\mathbb{L})$. The Stone spectrum $\mathcal{Q}(\mathbb{L})$ is dense in $\mathcal{D}(\mathbb{L})$ with respect to this topology.

Note that $\mathcal{D}(\mathbb{L})$ is in general **not a Hausdorff space**: let $b, c \in \mathbb{L}, b < c$. Then $H_c \in \mathcal{D}_a(\mathbb{L}) \iff c \leq a$, hence $H_b \in \mathcal{D}_a(\mathbb{L})$ and therefore H_b and H_c cannot be separated. We return to the case that \mathbb{L} is the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra \mathcal{R} , although most of our considerations also hold for an arbitrary orthocomplemented complete lattice.

We shall need the following simple

Lemma 2.1 $\forall P \in \mathcal{P}(\mathcal{R}) : H_P = \bigcap_{P \in \mathfrak{B}} \mathfrak{B} \quad (=: \bigcap \mathcal{Q}_P(\mathcal{R})).$

Let $A \in \mathcal{R}_{sa}$ with corresponding spectral family E_λ^A and observable function f_A . We extend f_A to a function $\mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ on the space $\mathcal{D}(\mathcal{R})$ of dual ideals of $\mathcal{P}(\mathcal{R})$ (and we denote this extension again by f_A) in a natural manner:

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_A(\mathcal{J}) := \inf\{\lambda \mid E_\lambda^A \in \mathcal{J}\}.$$

There are two fundamental properties of the function $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$. The first one is expressed in the following

Proposition 2.1 *Let $(\mathcal{J}_j)_{j \in J}$ be a family in $\mathcal{D}(\mathcal{R})$. Then*

$$f_A\left(\bigcap_{j \in J} \mathcal{J}_j\right) = \sup_{j \in J} f_A(\mathcal{J}_j).$$

The other is

Proposition 2.2 $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ *is upper semicontinuous, i.e.*

$$\forall \mathcal{J}_0 \in \mathcal{D}(\mathcal{R}) \forall \varepsilon > 0 \exists P \in \mathcal{J}_0 \forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f_A(\mathcal{J}) < f_A(\mathcal{J}_0) + \varepsilon.$$

Note that, even if A is a projection, $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ need not be continuous! Proposition 2.1 implies that f_A is *decreasing*, where $\mathcal{D}(\mathcal{R})$ is partially ordered by inclusion. Upper semicontinuity for decreasing functions $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ can be reformulated:

Proposition 2.3 *For any function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ the following two properties are equivalent:*

- (i) f *is upper semicontinuous and decreasing.*
- (ii) $\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) = \inf\{f(H_P) \mid P \in \mathcal{J}\}.$

A direct consequence of this, proposition 2.1 and of lemma 2.1 is

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_A(\mathcal{J}) = \inf_{P \in \mathcal{J}} \sup_{\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})} f_A(\mathfrak{B})$$

and, in particular, $f_A(\mathcal{D}(\mathcal{R})) = sp(A)$ for all $A \in \mathcal{R}_{sa}$.

Definition 2.3 *A function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is called an **abstract observable function** if it is upper semicontinuous and satisfies the intersection condition*

$$f\left(\bigcap_{j \in J} \mathcal{J}_j\right) = \sup_{j \in J} f(\mathcal{J}_j)$$

for all families $(\mathcal{J}_j)_{j \in J}$ in $\mathcal{D}(\mathcal{R})$.

The intersection condition implies that an abstract observable function is decreasing. Hence by proposition 2.3 the definition of abstract observable functions can be reformulated as follows:

Remark 2.2 *$f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function if and only if the following two properties hold for f :*

- (i) $\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) = \inf\{f(H_P) \mid P \in \mathcal{J}\},$
- (ii) $f\left(\bigcap_{j \in J} \mathcal{J}_j\right) = \sup_{j \in J} f(\mathcal{J}_j)$ for all families $(\mathcal{J}_j)_{j \in J}$ in $\mathcal{D}(\mathcal{R})$.

A direct consequence of the intersection condition is the following

Remark 2.3 *Let $\lambda \in im f$. Then the inverse image $f^{-1}(\lambda) \subseteq \mathcal{D}(\mathcal{R})$ has a minimal element \mathcal{J}_λ which is simply given by*

$$\mathcal{J}_\lambda = \bigcap \{\mathcal{J} \in \mathcal{D}(\mathcal{R}) \mid f(\mathcal{J}) = \lambda\}.$$

We will now show how one can recover the spectral family E^A of $A \in \mathcal{R}_{sa}$ from the observable function f_A . This gives us the decisive hint for the proof that to each abstract observable function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ there is a unique $A \in \mathcal{R}_{sa}$ with $f = f_A$.

Lemma 2.2 *Let $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an observable function and let E^A be the spectral family corresponding to A . If $\lambda \in im f$, then*

$$\mathcal{J}_\lambda = \{P \in \mathcal{P}(\mathcal{R}) \mid \exists \mu > \lambda : P \geq E_\mu^A\}.$$

$\mathcal{J}_\lambda = H_{E_\lambda^A}$ if and only if E^A is constant on some interval $[\lambda, \lambda + \delta]$. Moreover

$$E_\lambda^A = \inf \mathcal{J}_\lambda.$$

Theorem 2.5 *Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an abstract observable function. Then there is a unique $A \in \mathcal{R}_{sa}$ such that $f = f_A$.*

The **proof** proceeds in three steps. In the first step we construct from the abstract observable function f an increasing family $(E_\lambda)_{\lambda \in imf}$ in $\mathcal{P}(\mathcal{R})$ and show in a second step that this family can be extended to a spectral family in \mathcal{R} . Finally, in the third step, we show that the selfadjoint operator $A \in \mathcal{R}$ corresponding to that spectral family has observable function $f_A = f$ and that A is uniquely determined by f . We present here only the first two steps and omit the third because it is rather technical.

Step 1 Let $\lambda \in imf$ and let $\mathcal{J}_\lambda \in \mathcal{D}(\mathcal{R})$ be the smallest dual ideal such that $f(\mathcal{J}_\lambda) = \lambda$. In view of lemma 2.2 we have no choice than to define

$$E_\lambda := \inf \mathcal{J}_\lambda.$$

Lemma 2.3 *The family $(E_\lambda)_{\lambda \in imf}$ is increasing.*

Proof: Let $\lambda, \mu \in imf$, $\lambda < \mu$. Then

$$\begin{aligned} f(\mathcal{J}_\mu) &= \mu \\ &= \max(\lambda, \mu) \\ &= \max(f(\mathcal{J}_\lambda), f(\mathcal{J}_\mu)) \\ &= f(\mathcal{J}_\lambda \cap \mathcal{J}_\mu). \end{aligned}$$

Hence, by the minimality of \mathcal{J}_μ ,

$$\mathcal{J}_\mu \subseteq \mathcal{J}_\lambda \cap \mathcal{J}_\mu \subseteq \mathcal{J}_\lambda$$

and therefore $E_\lambda \leq E_\mu$. \square

Lemma 2.4 *f is monotonely continuous, i.e. if $(\mathcal{J}_j)_{j \in J}$ is an increasing net in $\mathcal{D}(\mathcal{R})$ then*

$$f\left(\bigcup_{j \in J} \mathcal{J}_j\right) = \lim_j f(\mathcal{J}_j).$$

Proof: Obviously $\mathcal{J} := \bigcup_{j \in J} \mathcal{J}_j \in \mathcal{D}(\mathcal{R})$. As f is decreasing, $f(\mathcal{J}) \leq f(\mathcal{J}_j)$ for all $j \in J$ and $(f(\mathcal{J}_j))_{j \in J}$ is a decreasing net of real numbers. Hence

$$f(\mathcal{J}) \leq \lim_j f(\mathcal{J}_j).$$

Let $\varepsilon > 0$. Because f is upper semicontinuous there is $P \in \mathcal{J}$ such that $f(\mathcal{I}) < f(\mathcal{J}) + \varepsilon$ for all $\mathcal{I} \in \mathcal{D}_P(\mathcal{R})$. Now $P \in \mathcal{J}_k$ for some $k \in J$ and therefore

$$\lim_j f(\mathcal{J}_j) \leq f(\mathcal{J}_k) < f(\mathcal{J}) + \varepsilon,$$

which shows that also $\lim_j f(\mathcal{J}_j) \leq f(\mathcal{J})$ holds. \square

Corollary 2.1 *The image of an abstract observable function is compact.*

Proof: Because $\{I\} \subseteq \mathcal{J}$ for all $\mathcal{J} \in \mathcal{D}(\mathcal{R})$ we have $f \leq f(\{I\})$ on $\mathcal{D}(\mathcal{R})$. If $\lambda, \mu \in \text{im} f$ and $\lambda < \mu$ then $\mathcal{J}_\mu \subseteq \mathcal{J}_\lambda$, hence $\bigcup_{\lambda \in \text{im} f} \mathcal{J}_\lambda$ is a dual ideal and therefore contained in a maximal dual ideal $\mathfrak{B} \in \mathcal{D}(\mathcal{R})$. This shows $f(\mathfrak{B}) \leq f$ on $\mathcal{D}(\mathcal{R})$ and consequently $\text{im} f$ is bounded. Let $\lambda \in \overline{\text{im} f}$. Then there is an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im} f$ converging to λ or there is a decreasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im} f$ converging to λ . In the first case we have $\mathcal{J}_{\mu_{n+1}} \subseteq \mathcal{J}_{\mu_n}$ for all $n \in \mathbb{N}$ and therefore for $\mathcal{J} := \bigcap_n \mathcal{J}_{\mu_n} \in \mathcal{D}(\mathcal{R})$

$$f(\mathcal{J}) = \sup_n f(\mathcal{J}_{\mu_n}) = \sup_n \mu_n = \lambda.$$

In the second case we have $\mathcal{J}_{\mu_n} \subseteq \mathcal{J}_{\mu_{n+1}}$ for all $n \in \mathbb{N}$ and therefore $\mathcal{J} := \bigcup_n \mathcal{J}_{\mu_n} \in \mathcal{D}(\mathcal{R})$. Hence

$$f(\mathcal{J}) = \lim_n f(\mathcal{J}_{\mu_n}) = \lim_n \mu_n = \lambda.$$

Therefore $\lambda \in \text{im} f$ in both cases, i.e. $\text{im} f$ is also closed. \square

Step 2 We will now extend $(E_\lambda)_{\lambda \in \text{im} f}$ to a spectral family $E^f := (E_\lambda)_{\lambda \in \mathbb{R}}$. In defining E^f we have of course in mind that the spectrum of the selfadjoint operator A corresponding to E^f should coincide with $\text{im} f$. This forces us to define E_λ for $\lambda \notin \text{im} f$ in the following way. For $\lambda \notin \text{im} f$ let

$$S_\lambda := \{\mu \in \text{im} f \mid \mu < \lambda\}.$$

Then we define

$$E_\lambda := \begin{cases} 0 & \text{if } S_\lambda = \emptyset \\ E_{\sup S_\lambda} & \text{otherwise.} \end{cases}$$

Note that $f(\{I\}) = \max \text{im} f$ and that $\mathcal{J}_{f(\{I\})} = \{I\}$.

Lemma 2.5 *E^f is a spectral family.*

Proof: The only remaining point to prove is that E^f is continuous from the right, i.e. that $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu$ for all $\lambda \in \mathbb{R}$. This is obvious if $\lambda \notin \text{im} f$ or if there is some $\delta > 0$ such that $] \lambda, \lambda + \delta[\cap \text{im} f = \emptyset$. Therefore we are left with the case that there is a strictly decreasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im} f$ converging to λ . For all $n \in \mathbb{N}$ we have $f(\mathcal{J}_{\mu_n}) > f(\mathcal{J}_\lambda)$ and therefore $\mathcal{J}_{\mu_n} \subseteq \mathcal{J}_\lambda$. Hence $\bigcup_n \mathcal{J}_{\mu_n} \subseteq \mathcal{J}_\lambda$ and

$$f\left(\bigcup_n \mathcal{J}_{\mu_n}\right) = \lim_n f(\mathcal{J}_{\mu_n}) = \lambda$$

implies $\bigcup_n \mathcal{J}_{\mu_n} = \mathcal{J}_\lambda$ by the minimality of \mathcal{J}_λ . If $P \in \mathcal{J}_\lambda$ then $P \in \mathcal{J}_{\mu_n}$ for some n and therefore $E_{\mu_n} \leq P$. This shows $\bigwedge_{\mu > \lambda} E_\mu \leq P$. As $P \in \mathcal{J}_\lambda$ is arbitrary we can conclude that $\bigwedge_{\mu > \lambda} E_\mu \leq E_\lambda$. The reverse inequality is obvious. \square

The theorem confirms that there is no difference between “abstract” and “concrete” observable functions and therefore we will speak generally of observable functions.

Let $\mathcal{P}_0(\mathcal{R})$ denote the set of nonzero projections in \mathcal{R} . We will now show that observable functions can be characterized as functions $\mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ that satisfy a “continuous join condition”. Note that for an arbitrary family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$ we have

$$\bigcap_{k \in \mathbb{K}} H_{P_k} = H_{\bigvee_{k \in \mathbb{K}} P_k}.$$

If $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function then the intersection property implies

$$f(H_{\bigvee_{k \in \mathbb{K}} P_k}) = \sup_{k \in \mathbb{K}} f(H_{P_k}).$$

This leads to the following

Definition 2.4 *A bounded function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is called completely increasing if*

$$r\left(\bigvee_{k \in \mathbb{K}} P_k\right) = \sup_{k \in \mathbb{K}} r(P_k)$$

for every family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$.

Note that it is sufficient to assume in the foregoing definition that r is bounded from below because $r(I)$ is an upper bound, in fact the maximum, for an arbitrary increasing function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$. Because of the natural bijection $P \mapsto H_P$ between $\mathcal{P}_0(\mathcal{R})$ and the set $\mathcal{D}_{pr}(\mathcal{R})$

of principle dual ideals of $\mathcal{P}(\mathcal{R})$ each observable function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ induces by restriction a completely increasing function r_f :

$$\forall P \in \mathcal{P}_0(\mathcal{R}) : r_f(P) := f(H_P).$$

Conversely, each completely increasing function on $\mathcal{P}_0(\mathcal{R})$ induces an observable function so that we get a one to one correspondence between observable functions and completely increasing functions.

Definition 2.5 *Let $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ be a completely increasing function. Then we define a function $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ by*

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_r(\mathcal{J}) := \inf_{P \in \mathcal{J}} r(P).$$

It is obvious that

$$\forall P \in \mathcal{P}_0(\mathcal{R}) : f_r(H_P) = r(P)$$

holds.

Proposition 2.4 *The function $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ induced by the completely increasing function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function.*

We have formulated theorem 2.5 and the characterization of observable functions by completely increasing functions in the category of von Neumann algebras. A simple inspection of the proofs shows that we have used the fact that the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra \mathcal{R} is a complete orthomodular lattice. Therefore we can translate theorem 2.5 to the category of complete orthomodular lattices in the following way:

Theorem 2.6 *Let \mathbb{L} be a complete orthomodular lattice and let $f : \mathcal{D}(\mathbb{L}) \rightarrow \mathbb{R}$ be an abstract observable function. Then there is a unique spectral family E in \mathbb{L} such that $f = f_E$.*

Finally, we like to present a characterization of those bounded continuous functions $f : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ that are observable.

Let $f : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ be a bounded continuous function. Because of

$$H_P = \bigcap \mathcal{Q}_P(\mathcal{R})$$

for all $P \in \mathcal{P}_0(\mathcal{R})$ it is natural to define

$$r(P) := \sup\{f(\mathfrak{B}) | \mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})\}.$$

Suppose that $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is completely increasing. (If \mathcal{R} is abelian, then this assumption is automatically satisfied.) Let $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be the corresponding observable function.

Lemma 2.6 *f coincides with the restriction of f_r to $\mathcal{Q}(\mathcal{R})$.*

Proof: We have to show that

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{R}) : f(\mathfrak{B}) = \inf_{P \in \mathfrak{B}} r(P)$$

holds.

From the definition of r we see that $f(\mathfrak{B}) \leq m := \inf_{P \in \mathfrak{B}} r(P)$. Let $\varepsilon > 0$. Because f is continuous there is $P_0 \in \mathfrak{B}$ such that $f(\mathfrak{C}) < f(\mathfrak{B}) + \varepsilon$ on $\mathcal{Q}_{P_0}(\mathcal{R})$. Hence

$$m \leq r(P_0) = \sup f(\mathcal{Q}_{P_0}(\mathcal{R})) \leq f(\mathfrak{B}) + \varepsilon.$$

This shows $m \leq f(\mathfrak{B})$. \square

Finally, we can formulate a criterion for the observability of a continuous function $f : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$:

Theorem 2.7 *Let \mathcal{R} be a von Neumann algebra. Then a bounded continuous function $f : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function if and only if the induced function*

$$\begin{array}{ccc} r_f : \mathcal{P}_0(\mathcal{R}) & \rightarrow & \mathbb{R} \\ P & \mapsto & \sup_{\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})} f(\mathfrak{B}) \end{array}$$

is completely increasing.

2.2 Classical Observables

In the previous subsection we have seen that selfadjoint elements A of a von Neumann algebra \mathcal{R} correspond to certain continuous real valued functions f_A on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of $\mathcal{P}(\mathcal{R})$.

Now we will show that continuous real valued functions on a Hausdorff space M can be described by spectral families with values in the complete lattice $\mathcal{T}(M)$ of open subsets of M . These spectral families $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ can be characterized abstractly by a certain property of the mapping σ . Thus also a classical observable has a “quantum mechanical” description. This shows that classical and quantum mechanical observables are on the same structural footing: either as functions or as spectral families. Similar results hold for functions on a set M that are measurable with respect to a σ -algebra of subsets of M . Due to our time limit, we confine ourselves to the statement of the main result.

We would like to begin with some simple examples:

Example 2.2 The following settings define spectral families $\sigma_{id}, \sigma_{abs}, \sigma_{ln}, \sigma_{step}$ in $\mathcal{T}(\mathbb{R})$:

$$\sigma_{id}(\lambda) :=] - \infty, \lambda[, \quad (2)$$

$$\sigma_{abs}(\lambda) :=] - \lambda, \lambda[\quad (3)$$

$$\sigma_{ln}(\lambda) :=] - \exp(\lambda), \exp(\lambda)[\quad (4)$$

$$\sigma_{step}(\lambda) :=] - \infty, [\lambda][\quad (5)$$

where $\lfloor \lambda \rfloor$ denotes the “floor of $\lambda \in \mathbb{R}$ ”:

$$\lfloor \lambda \rfloor = \max\{n \in \mathbb{Z} \mid n \leq \lambda\}.$$

The names of these spectral families sound somewhat crazy at the moment, but we will justify them soon.

In close analogy to the case of spectral families in the lattice $\mathbb{L}(\mathcal{H})$, each spectral family in $\mathcal{T}(M)$ induces a function on a subset of M .

Definition 2.6 Let $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ be a spectral family in $\mathcal{T}(M)$. Then

$$\mathcal{D}(\sigma) := \{x \in M \mid \exists \lambda \in \mathbb{R} : x \notin \sigma(\lambda)\}$$

is called the **admissible domain of σ** .

Note that

$$\mathcal{D}(\sigma) = M \setminus \bigcap_{\lambda \in \mathbb{R}} \sigma(\lambda).$$

Remark 2.4 The admissible domain $\mathcal{D}(\sigma)$ of a spectral family $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ is dense in M .

On the other hand it may happen that $\mathcal{D}(\sigma) \neq M$. The spectral family σ_{ln} is a simple example:

$$\forall \lambda \in \mathbb{R} : 0 \in \sigma_{ln}(\lambda).$$

Each spectral family $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ induces a function $f_\sigma : \mathcal{D}(\sigma) \rightarrow \mathbb{R}$:

Definition 2.7 Let $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ be a spectral family with admissible domain $\mathcal{D}(\sigma)$. Then the function $f_\sigma : \mathcal{D}(\sigma) \rightarrow \mathbb{R}$, defined by

$$\forall x \in \mathcal{D}(\sigma) : f_\sigma(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\},$$

is called the **function induced by σ** .

In complete analogy to the operator case we define the spectrum of a spectral family σ :

Definition 2.8 *Let $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ be a spectral family. Then*

$$R(\sigma) := \{\lambda \in \mathbb{R} \mid \sigma \text{ is constant on a neighborhood of } \lambda\}$$

*is called the **resolvent set** of σ , and*

$$sp(\sigma) := \mathbb{R} \setminus R(\sigma)$$

*is called the **spectrum** of σ .*

Obviously $sp(\sigma)$ is a closed subset of \mathbb{R} .

Proposition 2.5 *Let $f_\sigma : \mathcal{D}(\sigma) \rightarrow \mathbb{R}$ be the function induced by the spectral family $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$. Then*

$$sp(\sigma) = \overline{im f_\sigma}.$$

The functions induced by our foregoing examples are

$$f_{\sigma_{id}}(x) = x \tag{6}$$

$$f_{\sigma_{abs}}(x) = |x| \tag{7}$$

$$f_{\sigma_{ln}}(x) = \ln |x| \quad \text{and } \mathcal{D}(\sigma_{ln}) = \mathbb{R} \setminus \{0\} \tag{8}$$

$$f_{\sigma_{step}} = \sum_{n \in \mathbb{Z}} n \chi_{[n, n+1[} \tag{9}$$

There is a fundamental difference between the spectral families $\sigma_{id}, \sigma_{abs}, \sigma_{ln}$ on the one side and σ_{step} on the other. The function induced by σ_{step} is not continuous. This fact is mirrored in the spectral families: the first three spectral families have the property

$$\forall \lambda < \mu : \overline{\sigma(\lambda)} \subseteq \sigma(\mu).$$

Obviously σ_{step} fails to have this property.

Definition 2.9 *A spectral family $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ is called **continuous** if*

$$\forall \lambda < \mu : \overline{\sigma(\lambda)} \subseteq \sigma(\mu)$$

holds.

Remark 2.5 *The admissible domain $\mathcal{D}(\sigma)$ of a continuous spectral family $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ is an open (and dense) subset of M .*

Remark 2.6 *If $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ is a continuous spectral family, then for all $\lambda \in \mathbb{R}$ $\sigma(\lambda)$ is a regular open set, i.e. $\sigma(\lambda)$ is the interior of its closure.*

The importance of continuous spectral families becomes manifest in the following

Theorem 2.8 *Let M be a Hausdorff space. Then every continuous function $f : M \rightarrow \mathbb{R}$ induces a continuous spectral family $\sigma_f : \mathbb{R} \rightarrow \mathcal{T}(M)$ by*

$$\forall \lambda \in \mathbb{R} : \sigma_f(\lambda) := \text{int}(f^{-1}([\lambda, \infty])).$$

The admissible domain $\mathcal{D}(\sigma_f)$ equals M and the function $f_{\sigma_f} : M \rightarrow \mathbb{R}$ induced by σ_f is f . Conversely, if $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ is a continuous spectral family, then the function

$$f_\sigma : \mathcal{D}(\sigma) \rightarrow \mathbb{R}$$

induced by σ is continuous and the induced spectral family σ_{f_σ} in $\mathcal{T}(\mathcal{D}(\sigma))$ is the restriction of σ to the admissible domain $\mathcal{D}(\sigma)$:

$$\forall \lambda \in \mathbb{R} : \sigma_{f_\sigma}(\lambda) = \sigma(\lambda) \cap \mathcal{D}(\sigma).$$

One may wonder why we have defined the function, that is induced by a spectral family σ , on M and not on the Stone spectrum $\mathcal{Q}(\mathcal{T}(M))$. A quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{T}(M))$ is called *finite* if $\bigcap_{U \in \mathfrak{B}} \overline{U} \neq \emptyset$. If \mathfrak{B} is finite, then this intersection consists of a single element $x_{\mathfrak{B}} \in M$, and we call \mathfrak{B} a quasipoint *over* $x_{\mathfrak{B}}$. Note that for a compact space M , all quasipoints are finite. Moreover, one can show that for compact M , the mapping $pt : \mathfrak{B} \mapsto x_{\mathfrak{B}}$ from $\mathcal{Q}(\mathcal{T}(M))$ onto M is continuous and *identifying*.

Remark 2.7 *Let $\sigma : \mathbb{R} \rightarrow \mathcal{T}(M)$ be a continuous spectral family and let $x \in \mathcal{D}(\sigma)$. Then for all quasipoints $\mathfrak{B}_x \in \mathcal{Q}(\mathcal{T}(M))$ over x we have*

$$f_\sigma(\mathfrak{B}_x) = f_\sigma(x).$$

Therefore, if M is compact, it makes no difference whether we define f_σ in M or in $\mathcal{Q}(\mathcal{T}(M))$.

3 Presheaves Again: Quantum Observables as Global Sections

The abstract characterization of (quantum) observable functions leads to a natural definition of restricting selfadjoint elements of a von Neumann algebra \mathcal{R} to a subalgebra \mathcal{M} . Again we denote a completely increasing function on $\mathcal{P}_0(\mathcal{R})$ and the corresponding observable function (on $\mathcal{Q}(\mathcal{R})$ or $\mathcal{D}(\mathcal{R})$) by the same letter and speak simply of an observable function. We denote the set of observable functions for \mathcal{R} by $\mathcal{O}(\mathcal{R})$. Obviously we have

Remark 3.1 *Let \mathcal{M} be a von Neumann subalgebra of a von Neumann algebra \mathcal{R} and let $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ be an observable function. Then the restriction*

$$\varrho_{\mathcal{M}}f := f|_{\mathcal{P}_0(\mathcal{M})}$$

is an observable function for \mathcal{M} . It is called the restriction of f to \mathcal{M} .

This definition is absolutely natural. However, if A is a selfadjoint operator in \mathcal{R} then the observable function $f_A : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ corresponding to A is a rather abstract encoding of A . So before we proceed we will describe the restriction map

$$\begin{array}{ccc} \varrho_{\mathcal{M}} : \mathcal{O}(\mathcal{R}) & \rightarrow & \mathcal{O}(\mathcal{M}) \\ f_A & \mapsto & \varrho_{\mathcal{M}}f_A \end{array}$$

in terms of spectral families.

To this end we define

Definition 3.1 *Let \mathcal{F} be a filterbase in $\mathcal{P}_0(\mathcal{R})$. Then*

$$C_{\mathcal{R}}(\mathcal{F}) := \{Q \in \mathcal{P}_0(\mathcal{R}) \mid \exists P \in \mathcal{F} : P \leq Q\}$$

is called the cone over \mathcal{F} in \mathcal{R} .

Clearly $C_{\mathcal{R}}(\mathcal{F})$ is a dual ideal and it is easy to see that it is the *smallest dual ideal that contains \mathcal{F}* . A dual ideal $\mathcal{I} \in \mathcal{D}(\mathcal{M})$ is, in particular, a filterbase in $\mathcal{P}(\mathcal{R})$, so $C_{\mathcal{R}}(\mathcal{I})$ is well defined.

Proposition 3.1 *Let $f \in \mathcal{O}(\mathcal{R})$. Then*

$$(\varrho_{\mathcal{M}}f)(\mathcal{I}) = f(C_{\mathcal{R}}(\mathcal{I}))$$

for all $\mathcal{I} \in \mathcal{D}(\mathcal{M})$.

Definition 3.2 For a projection Q in \mathcal{R} let

$$c_{\mathcal{M}}(Q) := \bigvee \{P \in \mathcal{P}(\mathcal{M}) \mid P \leq Q\} \quad \text{and} \quad s_{\mathcal{M}}(Q) := \bigwedge \{P \in \mathcal{P}(\mathcal{M}) \mid P \geq Q\}.$$

$c_{\mathcal{M}}(Q)$ is called the \mathcal{M} -core, $s_{\mathcal{M}}(Q)$ the \mathcal{M} -support of Q .

The \mathcal{M} -support is a natural generalization of the notion of central support which is the \mathcal{M} -support if \mathcal{M} is the center of \mathcal{R} . Note that if $Q \notin \mathcal{M}$ then $c_{\mathcal{M}}(Q) < Q < s_{\mathcal{M}}(Q)$. The \mathcal{M} -core and the \mathcal{M} -support are related in a simple manner:

Remark 3.2 $c_{\mathcal{M}}(Q) + s_{\mathcal{M}}(I - Q) = I$ for all $Q \in \mathcal{P}(\mathcal{R})$.

Lemma 3.1 Let $E = (E_{\lambda})_{\lambda \in \mathbb{R}}$ be a spectral family in \mathcal{R} and for $\lambda \in \mathbb{R}$ define

$$(c_{\mathcal{M}}E)_{\lambda} := c_{\mathcal{M}}(E_{\lambda}), \quad (s_{\mathcal{M}}E)_{\lambda} := \bigwedge_{\mu > \lambda} s_{\mathcal{M}}(E_{\mu}).$$

Then $c_{\mathcal{M}}E := ((c_{\mathcal{M}}E)_{\lambda})_{\lambda \in \mathbb{R}}$ and $s_{\mathcal{M}}E := ((s_{\mathcal{M}}E)_{\lambda})_{\lambda \in \mathbb{R}}$ are spectral families in \mathcal{M} .

Proposition 3.2 Let $f \in \mathcal{O}(\mathcal{R})$ and let E be the spectral family corresponding to f . Then $c_{\mathcal{M}}E$ is the spectral family corresponding to $\varrho_{\mathcal{M}}f$.

Proof: Let \mathcal{I} be a dual ideal in $\mathcal{P}(\mathcal{M})$. Then $(\varrho_{\mathcal{M}}f)(\mathcal{I}) = f(C_{\mathcal{R}}(\mathcal{I}))$ and

$$\begin{aligned} f(C_{\mathcal{R}}(\mathcal{I})) &= \inf \{ \lambda \mid E_{\lambda} \in C_{\mathcal{R}}(\mathcal{I}) \} \\ &= \inf \{ \lambda \mid \exists P \in \mathcal{I} : P \leq E_{\lambda} \} \\ &= \inf \{ \lambda \mid c_{\mathcal{M}}(E_{\lambda}) \in \mathcal{I} \}. \end{aligned}$$

Thus the assertion follows from theorem 2.5. \square

By theorem 2.5, the restriction map $\varrho_{\mathcal{M}} : \mathcal{O}(\mathcal{R}) \rightarrow \mathcal{O}(\mathcal{M})$ induces a restriction map

$$\begin{aligned} \varrho_{\mathcal{M}} : \mathcal{R}_{sa} &\rightarrow \mathcal{M}_{sa} \\ A &\mapsto \varrho_{\mathcal{M}}A \end{aligned}$$

for selfadjoint operators. In particular, we obtain

Corollary 3.1 $\varrho_{\mathcal{M}}Q = s_{\mathcal{M}}(Q)$ for all projections Q in \mathcal{R} .

The corollary shows that the restriction map $\varrho_{\mathcal{M}} : \mathcal{R}_{sa} \rightarrow \mathcal{M}_{sa}$ has the important property that it maps projections to projections and acts as the identity on $\mathcal{P}(\mathcal{M})$. It also shows that in general $\varrho_{\mathcal{M}}$ is not linear: if $P, Q \in \mathcal{P}(\mathcal{R})$ such that $PQ = 0$ then it is possible that $s_{\mathcal{M}}(P)s_{\mathcal{M}}(Q) \neq 0$ and therefore $s_{\mathcal{M}}(P + Q) \neq s_{\mathcal{M}}(P) + s_{\mathcal{M}}(Q)$.

Proposition 3.2 and lemma 3.1 suggest still another natural possibility for defining a restriction map $\sigma_{\mathcal{M}} : \mathcal{R}_{sa} \rightarrow \mathcal{M}_{sa}$: if E^A is the spectral family corresponding to $A \in \mathcal{R}_{sa}$ then $\sigma_{\mathcal{M}}A$ is the selfadjoint operator defined by the spectral family $s_{\mathcal{M}}E^A$.

One can define a new partial order on \mathcal{R}_{sa} by

$$A \leq_s B \quad :\Longleftrightarrow \quad f_A \leq f_B.$$

On the level of spectral families, this can be expressed by

$$A \leq_s B \quad :\Longleftrightarrow \quad \forall \lambda \in \mathbb{R} : E_{\lambda}^B \leq E_{\lambda}^A.$$

This is the reason for calling \leq_s the *spectral order* on \mathcal{R}_{sa} . It is coarser than the usual one, but it has the advantage to turn \mathcal{R}_{sa} to a *boundedly complete lattice* ([3, 5]).

Proposition 3.3 *Let \mathcal{M} be a von Neumann subalgebra of the von Neumann algebra \mathcal{R} . Then, for all $A \in \mathcal{R}_{sa}$, we have*

$$\sigma_{\mathcal{M}}A = \bigvee \{B \in \mathcal{M}_{sa} \mid B \leq_s A\}$$

and

$$\varrho_{\mathcal{M}}A = \bigwedge \{C \in \mathcal{M}_{sa} \mid A \leq_s C\},$$

where $\sigma_{\mathcal{M}}A, \varrho_{\mathcal{M}}A$ are considered as elements of \mathcal{R} and \bigvee, \bigwedge denote the greatest lower bound and the least upper bound with respect to the spectral order.

This proposition shows that the two restriction mappings $\varrho_{\mathcal{M}}$ and $\sigma_{\mathcal{M}}$ from \mathcal{R}_{sa} onto \mathcal{M}_{sa} are on an equal footing. It is not difficult to determine the observable function $\sigma_{\mathcal{M}}f$ corresponding to $\sigma_{\mathcal{M}}A$, given the observable function f corresponding to $A \in \mathcal{R}_{sa}$.

The restrictions $\varrho_{\mathcal{M}}A$ and $\sigma_{\mathcal{M}}A$ can be seen as coarse grainings of A . We will demonstrate this in the abelian case: let \mathcal{A} be a von Neumann subalgebra of the abelian von Neumann algebra \mathcal{B} . We will show how the restriction maps $\varrho_{\mathcal{A}}^{\mathcal{B}} : \mathcal{B}_{sa} \rightarrow \mathcal{A}_{sa}$ and $\sigma_{\mathcal{A}}^{\mathcal{B}} : \mathcal{B}_{sa} \rightarrow \mathcal{A}_{sa}$ act on observable functions $f : \mathcal{Q}(\mathcal{B}) \rightarrow \mathbb{R}$ or, in other words, how the Gelfand transformation behaves with respect to the restrictions $\varrho_{\mathcal{A}}^{\mathcal{B}}$ and $\sigma_{\mathcal{A}}^{\mathcal{B}}$.

Proposition 3.4 *Let \mathcal{A}, \mathcal{B} be as above and let $f : \mathcal{Q}(\mathcal{B}) \rightarrow \mathbb{R}$ be an observable function. Then we have for all $\gamma \in \mathcal{Q}(\mathcal{A})$:*

$$(i) \ (\varrho_{\mathcal{A}}^{\mathcal{B}} f)(\gamma) = \sup\{f(\beta) \mid \gamma \subseteq \beta\} \text{ and}$$

$$(ii) \ (\sigma_{\mathcal{A}}^{\mathcal{B}} f)(\gamma) = \inf\{f(\beta) \mid \gamma \subseteq \beta\}.$$

Now consider three von Neumann subalgebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of \mathcal{R} such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$. Then the corresponding restriction maps $\varrho_{\mathcal{B}}^{\mathcal{C}} : \mathcal{C}_{sa} \rightarrow \mathcal{B}_{sa}$, $\varrho_{\mathcal{A}}^{\mathcal{B}} : \mathcal{B}_{sa} \rightarrow \mathcal{A}_{sa}$ and $\varrho_{\mathcal{A}}^{\mathcal{C}} : \mathcal{C}_{sa} \rightarrow \mathcal{A}_{sa}$ obviously satisfy

$$\varrho_{\mathcal{A}}^{\mathcal{C}} = \varrho_{\mathcal{A}}^{\mathcal{B}} \circ \varrho_{\mathcal{B}}^{\mathcal{C}} \quad \text{and} \quad \varrho_{\mathcal{A}}^{\mathcal{A}} = id_{\mathcal{A}_{sa}}. \quad (10)$$

The set $\mathfrak{S}(\mathcal{R})$ of all von Neumann subalgebras of \mathcal{R} is a lattice with respect to the partial order given by inclusion. The meet of $\mathcal{A}, \mathcal{B} \in \mathfrak{S}(\mathcal{R})$ is defined as the intersection,

$$\mathcal{A} \wedge \mathcal{B} := \mathcal{A} \cap \mathcal{B},$$

and the join as the subalgebra generated by \mathcal{A} and \mathcal{B} :

$$\mathcal{A} \vee \mathcal{B} := (\mathcal{A} \cup \mathcal{B})''.$$

The join is a rather intricate operation. This can already be seen in the most simple (non-trivial) example $\text{lin}_{\mathbb{C}}\{I, P\} \vee \text{lin}_{\mathbb{C}}\{I, Q\}$ for two non-commuting projections $P, Q \in \mathcal{R}$. Fortunately we don't need it really.

The subset $\mathfrak{A}(\mathcal{R}) \subseteq \mathfrak{S}(\mathcal{R})$ of all *abelian* von Neumann subalgebras of \mathcal{R} is also partially ordered by inclusion but it is only a *semilattice*: the meet of two (in fact of an arbitrary family of) elements of $\mathfrak{A}(\mathcal{R})$ always exists but the join does not in general. Both $\mathfrak{S}(\mathcal{R})$ and $\mathfrak{A}(\mathcal{R})$ have a smallest element, namely $\mathbb{O} := \mathbb{C}I$. However, unless \mathcal{R} is itself abelian, there is no greatest element in $\mathfrak{A}(\mathcal{R})$. Anyway, $\mathfrak{S}(\mathcal{R})$ and $\mathfrak{A}(\mathcal{R})$ can be considered as the sets of objects of (small) *categories* whose morphisms are the inclusion maps.

In quantum physics the (maximal) abelian von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$ are called *contexts*. We generalize this notion in the following

Definition 3.3 *The small category $\mathcal{CON}(\mathcal{R})$, whose objects are the abelian von Neumann subalgebras of \mathcal{R} and whose morphisms are the inclusion maps, is called the **context category of the von Neumann algebra \mathcal{R}** .*

We define a *presheaf* $\mathcal{O}_{\mathcal{R}}$ on the context category $\mathcal{CON}(\mathcal{R})$ of \mathcal{R} by sending objects $\mathcal{A} \in \mathfrak{A}(\mathcal{R})$ to $\mathcal{O}_{\mathcal{R}}(\mathcal{A}) := \mathcal{A}_{sa}$ (or equivalently to $\mathcal{O}(\mathcal{A})$) and morphisms $\mathcal{A} \hookrightarrow \mathcal{B}$ to restrictions $\varrho_{\mathcal{A}}^{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{A}$. This gives a contravariant functor, i.e. a presheaf on $\mathcal{CON}(\mathcal{R})$.

Definition 3.4 *The presheaf $\mathcal{O}_{\mathcal{R}}$ is called the **observable presheaf** of the von Neumann algebra \mathcal{R} .*

Every observable function $f \in \mathcal{O}(\mathcal{R})$ induces a family $(f_{\mathcal{A}})_{\mathcal{A} \in \mathfrak{A}(\mathcal{R})}$ of observable functions $f_{\mathcal{A}} \in \mathcal{O}(\mathcal{A})$, defined by $f_{\mathcal{A}} := \varrho_{\mathcal{A}}^{\mathcal{R}} f$. This family has the following compatibility property:

$$\forall \mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{R}) : \varrho_{\mathcal{A} \cap \mathcal{B}}^{\mathcal{A}} f_{\mathcal{A}} = \varrho_{\mathcal{A} \cap \mathcal{B}}^{\mathcal{B}} f_{\mathcal{B}}. \quad (11)$$

$(f_{\mathcal{A}})_{\mathcal{A} \in \mathfrak{A}(\mathcal{R})}$ is therefore a *global section* of the presheaf $\mathcal{O}_{\mathcal{R}}$ in the following general sense.

Definition 3.5 *Let \mathbf{C} be a category and $\mathcal{S} : \mathbf{C} \rightarrow \mathbf{Set}$ a presheaf, i.e. a contravariant functor from \mathbf{C} to the category \mathbf{Set} of sets. A global section of \mathcal{S} assigns to every object a of \mathbf{C} an element $\sigma(a)$ of the set $\mathcal{S}(a)$ such that for every morphism $\varphi : b \rightarrow a$ of \mathbf{C}*

$$\sigma(b) = \mathcal{S}(\varphi)(\sigma(a))$$

holds.

In the case of the observable presheaf $\mathcal{O}_{\mathcal{R}}$ there are plenty of global sections because each $A \in \mathcal{R}_{sa}$ induces one. Here the natural question arises whether all global sections of $\mathcal{O}_{\mathcal{R}}$ are induced by selfadjoint elements of \mathcal{R} . This is certainly not true if the Hilbert space \mathcal{H} has dimension two. For in this case the constraints 11 are void and therefore *any* function on the complex projective line defines a global section of $\mathcal{O}_{\mathcal{L}(\mathcal{H})}$. But Gleason's (or Kochen-Specker's) theorem teaches us that the dimension two is something peculiar. One can show, however, that the phenomenon, that there are global sections of $\mathcal{O}_{\mathcal{R}}$ that are not induced by selfadjoint elements of \mathcal{R} , is not restricted to the dimension two.

Therefore, if one takes contextuality in quantum physics serious, it is natural to generalize the notion of quantum observable:

Definition 3.6 *Let \mathcal{R} be a von Neumann algebra. The global sections of the observable presheaf $\mathcal{O}_{\mathcal{R}}$ are called **contextual observables**.*

Contextual observables can be characterized as certain functions on $\mathcal{P}_0(\mathcal{R})$:

Proposition 3.5 *Let \mathcal{R} be a von Neumann algebra. There is a one-to-one correspondence between global sections of the observable presheaf $\mathcal{O}_{\mathcal{R}}$ and functions $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ that satisfy*

(i) $f(\bigvee_{k \in \mathbb{K}} P_k) = \sup_{k \in \mathbb{K}} f(P_k)$ for all **commuting** families $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$,

(ii) $f|_{\mathcal{P}_0(\mathcal{R}) \cap \mathcal{A}}$ is bounded for all $\mathcal{A} \in \mathfrak{A}(\mathcal{R})$.

Proof: Let $(f_{\mathcal{A}})_{\mathcal{A} \in \mathfrak{A}(\mathcal{R})}$ be a global section of $\mathcal{O}_{\mathcal{R}}$. Then the functions $f_{\mathcal{A}} : \mathcal{P}_0(\mathcal{R}) \cap \mathcal{A} \ (\mathcal{A} \in \mathfrak{A}(\mathcal{R}))$ can be glued to a function $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$:

Let $P \in \mathcal{P}_0(\mathcal{R})$ and let \mathcal{A} be an abelian von Neumann subalgebra of \mathcal{R} that contains P . Then

$$f(P) := f_{\mathcal{A}}(P)$$

does not depend on the choice of \mathcal{A} . Indeed, if $P \in \mathcal{A} \cap \mathcal{B}$, then $f_{\mathcal{A}}(P) = f_{\mathcal{B}}(P)$ by the compatibility property of global sections. It is obvious that f satisfies properties (i) and (ii).

If, conversely, a function $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ with the properties (i) and (ii) is given and if \mathcal{A} is an abelian von Neumann subalgebra of \mathcal{R} , then $f_{\mathcal{A}} := f|_{\mathcal{P}_0(\mathcal{R}) \cap \mathcal{A}}$ is a completely increasing function. The family $(f_{\mathcal{A}})_{\mathcal{A} \in \mathfrak{A}(\mathcal{R})}$ is then, by construction, a global section of $\mathcal{O}_{\mathcal{R}}$. \square

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